Lec 8: 02/13/2012

Fermi Acceleration (Cont'd):

We now turn to an alternative geometry in which the scattering of test particles occurs within a converging flow, arising across a shock. As we will see, the second-order process will turn into a first-order one in this case.

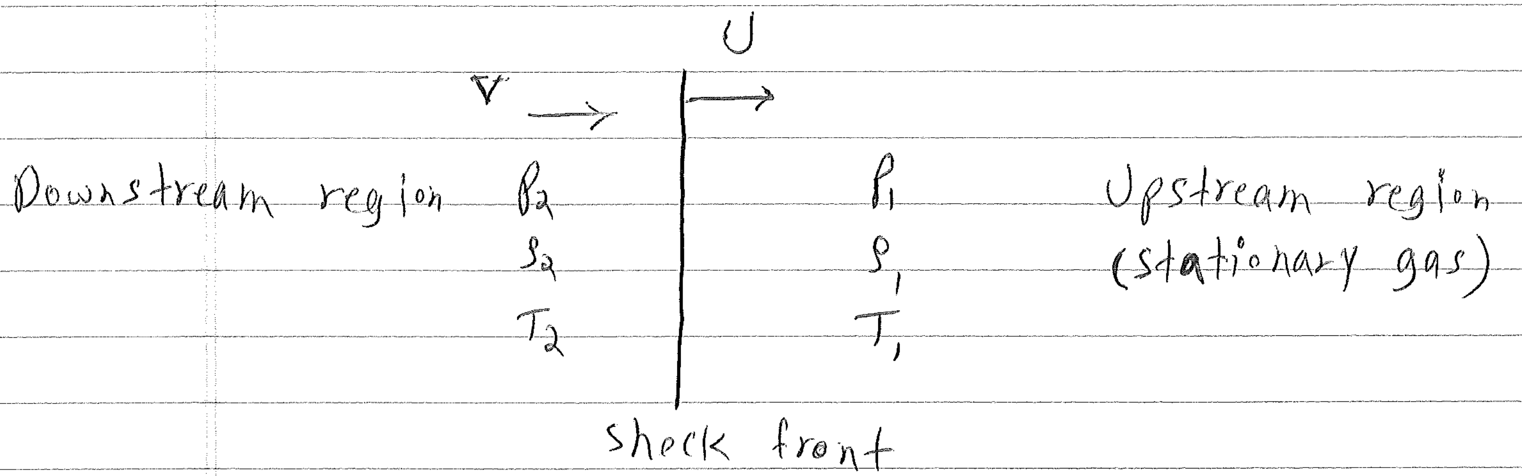
Shock waves are found ubiquitously in high-energy astrophysics and play a key role in many different astrophysical environments. It is a general property of perturbations in a gas that they are propagated away from their source at the speed of sound in the medium, which is given by:

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{ad}} = \sqrt{\frac{\gamma P}{\rho}} \qquad \left(\gamma = \frac{c_p}{c_v}\right)$$

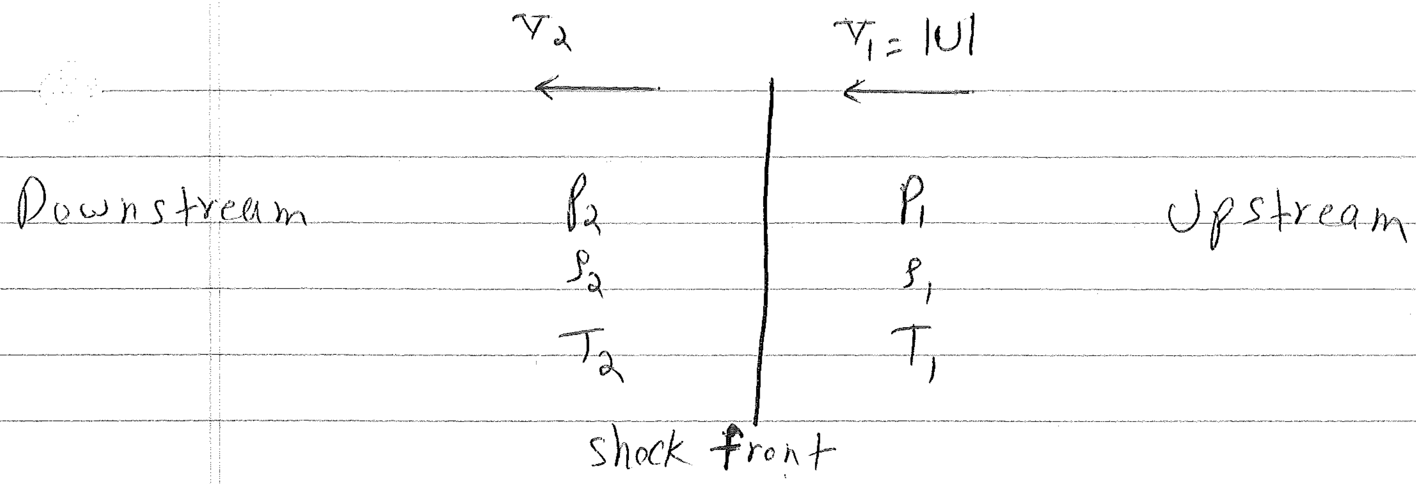
Here "ad" refers to the adiabatic condition (as opposed to isothermal).

If the source itself moves at a velocity greater than c_s, then the distrurbance cannot behave like a sound wave at all. There will be a discontinuity between the regions behind and ahead of the disturbance. These discontinuities are called shock waves.

Let us focus on plane shock waves and assume abrupt discontinuity between the two regions of fluid flow:



It is convenient to transform to a reference frame moving at velocity U, in which the shock front is stationary. In this frame we have:



The behaviour of the gas on passing through the shock front is described by a set of conservation relations. First, mass is conserved on passing through the discontinuity:

$$\rho_1 v_1 = \rho_2 v_2$$

Second, the energy flux is continuous. The energy flux through the shock front is $\rho_1 v_1 (\frac{1}{2} v_1^2 + w_1)$ and $\rho_2 v_2 (\frac{1}{2} v_2^2 + w_2)$ in the upstream and downstream regions respectively. Here $w = PV + \epsilon$ is the enthalpy per unit mass, where ϵ is the internal energy per unit mass and $V = \rho^{-1}$ is the specific volume.

Note that $\frac{1}{2} v^2$ term, which also shows up in the Bernoulli's equation, arises because of the net flow of the gas. The

Conservation of energy flux therefore implies:

$$\rho_1 v_1 \left(\frac{1}{2} v_1^2 + w_1\right) = \rho_2 v_2 \left(\frac{1}{2} v_2^2 + w_2\right)$$

Finally, the momentum flux through the shock front should be continuous. This results in:

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$$

For an ideal gas $w = \frac{\gamma P V}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}$. The mass flux is denoted by $j = \rho_1 v_1 = \rho_2 v_2$. We then find:

$$j^2 = \frac{P_2 - P_1}{V_1 - V_2} \qquad (V_1 = \rho_1^{-1},\ V_2 = \rho_2^{-1})$$

In addition:

$$v_1 - v_2 = j(V_1 - V_2) = \left[(P_2 - P_1)(V_1 - V_2)\right]^{\frac{1}{2}}$$

From the conservation of energy flux we obtain:

$$(w_1 - w_2) + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0$$

Using the relation $w = \frac{\gamma P V}{\gamma - 1}$ for an ideal gas, this leads to:

$$\frac{V_2}{V_1} = \frac{P_1(\gamma+1) + P_2(\gamma-1)}{P_1(\gamma-1) + P_2(\gamma+1)}$$

One can show that:

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

Here M_1 is the Mach number in the upstream region:

$$M_1 = \frac{v_1}{c_1} \quad , \quad c_1 = \left(\frac{\gamma P_1}{\rho_1}\right)^{\frac{1}{2}}$$

The density ratio is then found to be:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma-1)P_1 + (\gamma+1)P_2}{(\gamma+1)P_1 + (\gamma-1)P_2} = \frac{\gamma+1}{(\gamma-1) + \frac{2}{M_1^2}}$$

For very strong shocks $v_1 \gg c_1$, which results in:

$$\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1} \Rightarrow v_1 = 4 v_2 \Rightarrow v_2 = \frac{|U|}{4}$$

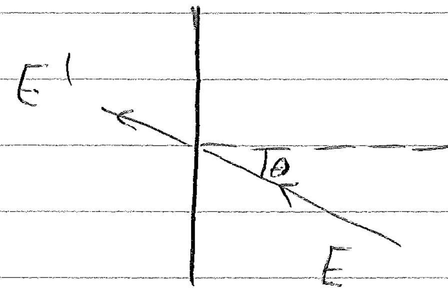
$\searrow$ $\gamma = \frac{5}{3}$ for an ideal monatomic gas

In the rest frame of upstream region, gas in the downstream region moves at a speed $\frac{3U}{4}$ behind the shock front.

Let us now discuss the actual process of particle acceleration.

A particle crossing from the upstream to downstream sides of the shock acquires an increase in its energy according to:

$E' = \gamma\left(E + \frac{3}{4} U P_\eta\right)$



$\gamma = \left[1 - \left(\frac{3U}{4c}\right)^2\right]^{-\frac{1}{2}} \simeq 1$, $P_\eta = \frac{E}{c}\cos\theta$

Here we have assumed a relativistic

particle $E = Pc$, and a non-relativistic shock $U \ll c$.

The energy increase is:

$\Delta E = E' - E \Rightarrow \frac{\Delta E}{E} \simeq \frac{3}{4}\frac{U}{c}\cos\theta$

Integrating over the incident angle $0 \leq \theta \leq \frac{\pi}{2}$, we find:

$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{3}{4}\frac{U}{c}\int_0^{\frac{\pi}{2}} P_{(\theta)} \cos\theta \, d\theta$, $P_{(\theta)} = 2\sin\theta\cos\theta \, d\theta$

Thus:

$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{1}{2}\frac{U}{c}$

After crossing the shock front, the particle's velocity vector

is randomized through elastic scattering off the gas

particles in the downstream region. The particle gains another fractional increase of $\frac{1}{2}\frac{U}{c}$ as it crosses the shock front back to the upstream region. This is the main difference with the simple one-dimensional example we considered earlier, in which the linear terms in head-on and catch-up collisions had different signs and hence cancelled out.

The average increase in energy per round trip is:

$$\beta = 1 + \frac{1}{2}\frac{U}{c} + \frac{1}{2}\frac{U}{c} = 1 + \frac{U}{c}$$

Now, we need to work out the probability P that the particle crosses the shock front back to the upstream region. According to kinetic theory, the flux of particles crossing a surface is $\frac{1}{4}nc$, where n is the number density of particles and particles are assumed to be relativistic.

Note that the flux represents the number of particles that cross the surface at incident angles $0 \leq \theta \leq \frac{\pi}{2}$ without

Collisions with other particles. That is, the particles are within one mean free path from the surface.

In the case of the shock, however, the front moves at a velocity $\frac{1}{4}U$. This leads to an effective increase in the distance that the particles must move before crossing the front. The average increase is $\frac{\frac{1}{4}U}{\frac{1}{4}C} = \frac{U}{C}$, which implies the probability P for the particles to cross the shock front back is going to be,

$$P = 1 - \frac{U}{C}$$

Having found P and β, we can now derive the dependence of the spectrom on E. Starting with an initial energy E_0 and number N_0 in the upstream region, after K Collisions we have:

$$E = E_0 \beta^K \quad , \quad N = N_0 P^K$$

Thus;

$$\ln\left(\frac{E}{E_0}\right) = k \ln \beta \quad , \quad \ln\left(\frac{N}{N_0}\right) = k \ln \rho$$

This results in:

$$\frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\frac{\ln \rho}{\ln \beta}}$$

Now:

$$\frac{\ln \rho}{\ln \beta} = \frac{\ln\left(1 - \frac{U}{c}\right)}{\ln\left(1 + \frac{U}{c}\right)} \approx \frac{-\frac{U}{c}}{\frac{U}{c}} = -1$$

Here we have used the fact that the shock is non-relativistic ($\frac{U}{c} << 1$). Therefore:

$$\frac{N}{N_0} \approx \left(\frac{E}{E_0}\right)^{-1} \Rightarrow \underbrace{N(E)\, d_E}_{dN} \propto E^{-2}\, d_E \Rightarrow N(E) \propto E^{-2}$$

The value of the exponent of the differential energy spectrum is close to the "observed" universal value of -2.5, which is remarkable.